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The state complexity of star-complement-star

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Abstract. We resolve an open question by determining matching (asymptotic) upper and lower bounds on the state complexity of the operation that sends a language L to $\left(\overline{L^*}\right)^*$.

1 Introduction

Let Σ be a finite nonempty alphabet, let $L \subseteq \Sigma^*$ be a language, let $\overline{L} = \Sigma^* - L$ denote the complement of L, and let L^* (resp., L^+) denote the Kleene closure (resp., positive closure) of the language L. If L is a regular language, its *state complexity* is defined to be the number of states in the minimal deterministic finite automaton accepting L [7]. In this paper we resolve an open question by determining matching (asymptotic) upper and lower bounds on the deterministic state complexity of the operations

$$L \to \left(\overline{L^*}\right)^*$$
$$L \to \left(\overline{L^+}\right)^+.$$

To simplify the exposition, we will write everything using an exponent notation, using c to represent complement, as follows:

$$L^{+c} := \overline{L^+}$$
$$L^{+c+} := (\overline{L^+})^+,$$

and similarly for L^{*c} and L^{*c*}.

Note that

$$L^{*c*} = \begin{cases} L^{+c+}, & \text{if } \varepsilon \notin L; \\ L^{+c+} \cup \{\varepsilon\}, & \text{if } \varepsilon \in L. \end{cases}$$

It follows that the state complexity of L^{+c+} and L^{*c*} differ by at most 1. In what follows, we will work only with L^{+c+}.

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2 Upper Bound

Consider a deterministic finite automaton (DFA) $D = (Q_n, \Sigma, \delta, 0, F)$ accepting a language L, where $Q_n := \{0, 1, \ldots, n-1\}$. As an example, consider the three-state DFA over $\{a, b, c, d\}$ shown in Fig. 1 (left). To get a nondeterministic finite automaton (NFA) N_1 for the language L^+ from the DFA D, we add an ε-transition from every non-initial final state to the state 0. In our example, we add an ε-transition from state 1 to state 0; see Fig. 1 (right). After applying the subset construction to the NFA N_1 we get a DFA D_1 for the language L^+. The state set of D_1 consists of subsets of Q_n see Fig. 2 (left). Here the sets in the labels of states are written without commas and brackets; thus, for example 012 stands for the set $\{0, 1, 2\}$. Next, we interchange the roles of the final and non-final states of the DFA D_1, and get a DFA D_2 for the language L^{+c}; see Fig. 2 (right).

To get an NFA N_3 for L^{+c+} from the DFA D_2, we add an ε-transition from each non-initial final state of D_2 to the state $\{0\}$, see Fig. 3 (top). Applying the subset construction to the NFA N_3 results in a DFA D_3 for the language L^{+c+} with its state set consisting of some sets of subsets of Q_n; see Fig. 3 (middle). Here, for example, the label $0, 2$ corresponds to the set $\{\{0\}, \{2\}\}$. This gives an upper bound of 2^{2^n} on the state complexity of the operation plus-complement-plus.

Our first result shows that in the minimal DFA for L^{+c+} we do not have any state $\{S_1, S_2, \ldots, S_k\}$, in which a set S_i is a subset of some other set S_j; see Fig. 3 (bottom). This reduces the upper bound to the number of antichains of subsets of an n-element set known as the Dedekind number $M(n)$ with [2]

$$\binom{n}{\lfloor n/2 \rfloor} \leq \log M(n) \leq \binom{n}{\lfloor n/2 \rfloor}\Big(1 + O(\frac{\log n}{n})\Big).$$

Fig. 1. DFA D for a language L and NFA N_1 for the language L^+.

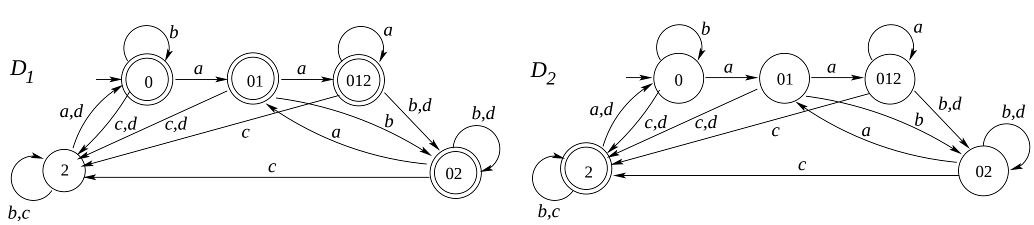

Fig. 2. DFA D_1 for language L^+ and DFA D_2 for the language L^{+c}.

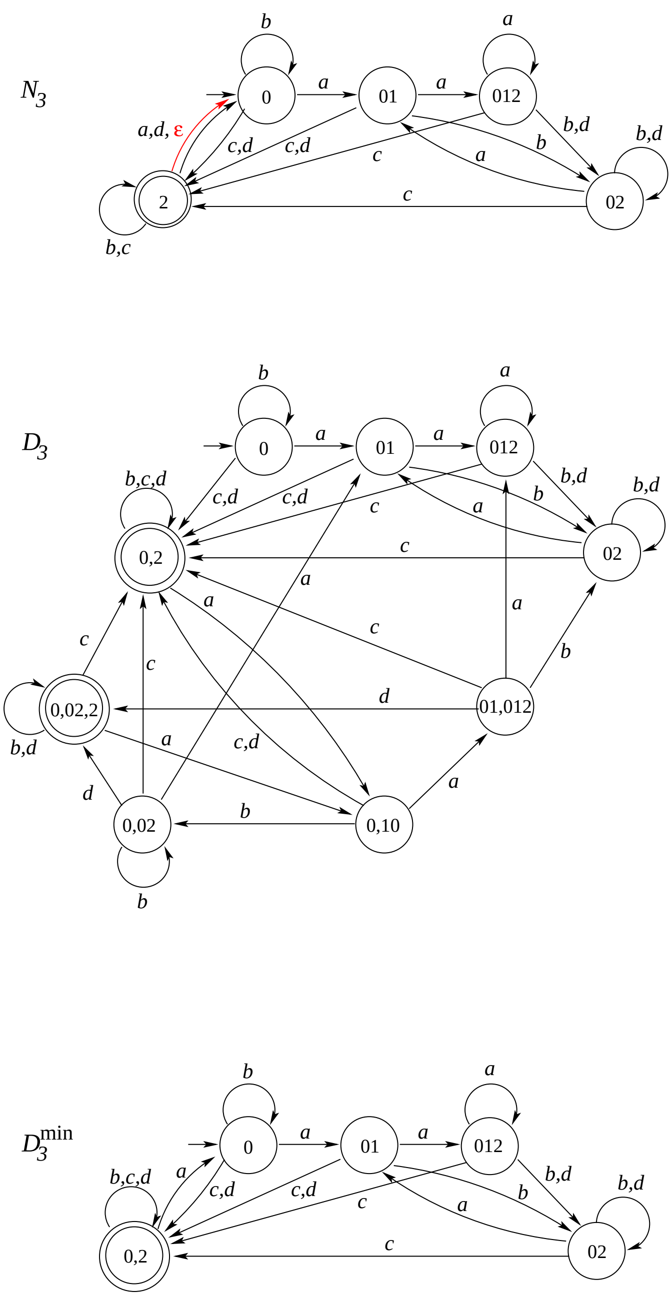

Fig. 3. NFA N_3, DFA D_3, and the minimal DFA $D_3^{\min}$ for the language L^{+c+}.

Lemma 1. *If S and T are subsets of Q_n such that $S \subseteq T$, then the states $\{S, T\}$ and $\{S\}$ of the DFA D_3 for the language L^{+c+} are equivalent.*

Proof. Let S and T be subsets of Q_n such that $S \subseteq T$. We only need to show that if a string w is accepted by the NFA N_3 starting from the state T, then it also is accepted by N_3 from the state S.

Assume w is accepted by N_3 from T. Then in the NFA N_3, an accepting computation on w from state T looks like this:

$$T \xrightarrow{u} T_1 \xrightarrow{\varepsilon} \{0\} \xrightarrow{v} T_2,$$

where $w = uv$, and state T goes to an accepting state T_1 on u without using any ε-transitions, then T_1 goes to $\{0\}$ on ε, and then $\{0\}$ goes to an accepting state T_2 on v; it also may happen that $w = u$, in which case the computation ends in T_1. Let us show that S goes to an accepting state of the NFA N_3 on u.

Since T goes to an accepting state T_1 on u in the NFA N_3 without using any ε-transition, state T goes to the accepting state T_1 in the DFA D_2, and therefore to the rejecting state T_1 of the DFA D_1. Thus, every state q in T goes to rejecting states in the NFA N_1. Since $S \subseteq T$, every state in S goes to rejecting states in the NFA N_1, and therefore S goes to a rejecting state S_1 in the DFA D_1, thus to the accepting state S_1 in the DFA D_2. Hence $w = uv$ is accepted from S in the NFA N_3 by computation

$$S \xrightarrow{u} S_1 \xrightarrow{\varepsilon} \{0\} \xrightarrow{v} T_2.$$

□

Hence whenever a state $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ of the DFA D_3 contains two subsets S_i and S_j with $i \neq j$ and $S_i \subseteq S_j$, then it is equivalet to state $\mathcal{S} \setminus \{S_j\}$. Using this property, we get the following result.

Lemma 2. *Let D be a DFA for a language L with state set Q_n, and $D_3^{\min}$ be the minimal DFA for L^{+c+} as described above. Then every state of $D_3^{\min}$ can be expressed in the form*

$$\mathcal{S} = \{X_1, X_2, \ldots, X_k\} \tag{1}$$

where

- *$1 \leq k \leq n$;*
- *there exist subsets $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k \subseteq Q_n$; and*
- *there exist $q_1, \ldots, q_k$, pairwise distinct states of D not in S_k; such that*
- *$X_i = \{q_i\} \cup S_i$ for $i = 1, 2, \ldots, k$.*

Proof. Let $D = (Q_n, \Sigma, \delta, 0, F)$.

For a state q in Q_n and a symbol a in Σ, let $q.a$ denote the state in Q_n, to which q goes on a, that is, $q.a = \delta(q, a)$. For a subset X of Q_n let $X.a$ denote the set of states to which states in X go by a, that is,

$$X.a = \bigcup_{q \in X} \{\delta(q, a)\}.$$

Consider transitions on a symbol a in automata D, N_1, D_1, D_2, N_3; Fig. 4 illustrates these transitions. In the NFA N_1, each state q goes to a state in $\{0, q.a\}$ if $q.a$ is a final state of D, and to state $q.a$ if $q.a$ is non-final. It follows that in the DFA D_1 for L^+, each state X (a subset of Q_n) goes on a to final state $\{0\} \cup X.a$ if $X.a$ contains a final state of D, and to non-final state $X.a$ if all states in $X.a$ are non-final in D. Hence in the DFA D_2 for L^{+c}, each state X goes on a to non-final state $\{0\} \cup X.a$ if $X.a$ contains a final state of D, and to the final state $X.a$ if all states in $X.a$ are non-final in D.

Therefore, in the NFA N_3 for L^{+c+}, each state X goes on a to a state in $\{\{0\}, X.a\}$ if all states in $X.a$ are non-final in D, and to state $\{0\} \cup X.a$ if $X.a$ contains a final state of D.

To prove the lemma for each state, we use induction on the length of the shortest path from the initial state to the state of $D_3^{\min}$ in question. The base case is a path of length 0. In this case, the initial state is $\{\{0\}\}$, which is in the required form (1) with $k = 1, q_1 = 0$, and $S_1 = \emptyset$.

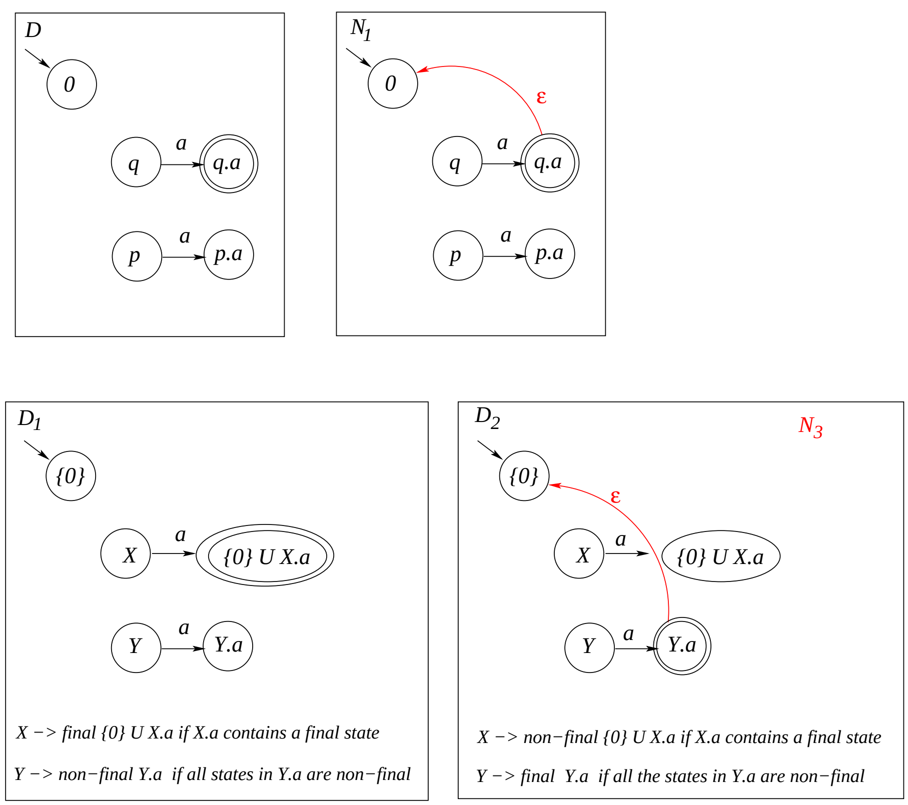

Fig. 4. Transitions under symbol a in automata D, N_1, D_1, D_2, N_3.

For the induction step, let

$$\mathcal{S} = \{X_1, X_2, \dots, X_k\},$$

where $1 \le k \le n$, and

- $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k \subseteq Q_n$,
- $q_1, \dots, q_k$ are pairwise distinct states of D that are not in S_k and
- $X_i = \{q_i\} \cup S_i$ for $i = 1, 2, \dots, k$.

We now prove the result for all states reachable from $\mathcal{S}$ on a symbol a.

First, consider the case that each X_i goes on a to a non-final state X_i' in the NFA N_3. It follows that $\mathcal{S}$ goes on a to $\mathcal{S}' = \{X_1', X_2', \dots, X_k'\}$, where

$$X_i' = \{q_i.a\} \cup S_i.a \cup \{0\}.$$

Write $p_i = q_i.a$ and $P_i = S_i.a \cup \{0\}$. Then we have $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_k \subseteq Q_n$.

If $p_i = p_j$ for some i, j with $i < j$, then $X_i' \subseteq X_j'$, and therefore X_j' can be removed from state $\mathcal{S}'$ in the minimal DFA $D_3^{\min}$. After several such removals, we arrive at an equivalent state

$$\mathcal{S}'' = \{X_1'', X_2'', \dots, X_\ell''\}$$

where $\ell \le k$, $X_i'' = \{r_i\} \cup R_i$ and the states $r_1, r_2, \dots, r_\ell$ are pairwise distinct.

If $r_i \in R_\ell$ for some i with $i < \ell$, then $X_i \subseteq R_\ell$; thus R_ℓ can be removed. After all such removals, we get an equivalent set

$$\mathcal{S}''' = \{X_1''', X_2''', \dots, X_m'''\}$$

where $m \le \ell$, $X_i''' = \{t_i\} \cup T_i$ and the states $t_1, t_2, \dots, t_m$ are pairwise distinct and $t_1, t_2, \dots, t_{m-1}$ are not in T_m. If $t_m \notin T_m$, then the state $\mathcal{S}'''$ is in the required form (1). Otherwise, if T_{m-1} is a proper subset of T_m, then there is a state t in $T_m - T_{m-1}$, and then we can take $X_m''' = \{t\} \cup T_m - \{t\}$: since $t_1, \dots, t_{m-1}$ are not in T_m, they are distinct from t, and moreover $T_{m-1} \subseteq T_m - \{t\}$.

If $T_{m-1} = T_m$, then $X_{m-1}''' \supseteq X_m'''$, and therefore X_{m-1}''' can be removed from $\mathcal{S}'''$. After all these removals we either reach some T_i that is a proper subset of T_m, and then pick a state t in $T_m - T_i$ in the same way as above, or we only get a single set T_m, which is in the required form $\{r_m\} \cup T_m - \{r_m\}$.

This proves that if each X_i in $\mathcal{S}$ goes on a to a non-final state X_i' in the NFA N_3, then $\mathcal{S}$ goes on a in the DFA $D_3^{\min}$ to a set that is in the required form (1).

Now consider the case that at least one X_j in $\mathcal{S}$ goes to a final state X_j' in the NFA N_3. It follows that $\mathcal{S}$ goes to a final state

$$\mathcal{S}' = \{\{0\}, X_1', X_2', \dots, X_k'\},$$

where $X_j' = \{q_j.a\} \cup S_j.a$ and if $i \ne j$, then $X_i' = \{q_i.a\} \cup S_i.a$ or $X_i' = \{0\} \cup \{q_i.a\} \cup S_i.a$ We now can remove all X_i that contain state 0, and arrive at an equivalent state

$$\mathcal{S}'' = \{\{0\}, X_1'', X_2'', \dots, X_\ell''\},$$

where $\ell \le k$, and $X_i'' = \{p_i\} \cup P_i$, and $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_\ell \subseteq Q_n$, and each p_i is distinct from 0.

Now in the same way as above we arrive at an equivalent state

$$\{\{0\}, \{t_1\} \cup T_1, \ldots, \{t_m\} \cup T_m\}$$

where $m \le \ell$, all the t_i are pairwise distinct and different from 0, and moreover, the states $t_1, \ldots, t_{m-1}$ are not in T_m. If t_m is not in T_m, then we are done. Otherwise, we remove all sets with $T_i = T_m$. We either arrive at a proper subset T_j of T_m, and may pick a state t in $T_m - T_j$ to play the role of new t_m, or we arrive at $\{\{0\}, T_m\}$, which is in the required form $\{\{0\} \cup \emptyset, t_m \cup T_m - \{t_m\}\}$. This completes the proof of the lemma. □

Corollary 1 (Star-Complement-Star: Upper Bound). *If a language L is accepted by a DFA of n states, then the language L^{*c*} is accepted by a DFA of $2^{O(n \log n)}$ states.*

Proof. Lemma 2 gives the following upper bound

$$\sum_{k=1}^{n} \binom{n}{k} k!(k+1)^{n-k}$$

since we first choose any permutation of k distinct elements $q_1, \ldots, q_k$, and then represent each set S_i as disjoint union of sets $S_1', S_2', \ldots, S_i'$ given by a function f from $Q_n - \{q_1, \ldots, q_k\}$ to $\{1, 2, \ldots, k+1\}$ as follows:

$$S_i' = \{q \mid f(q) = i\}, \qquad S_i = S_1' \mathbin{\dot\cup} S_2' \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} S_i',$$

while states with $f(q) = k+1$ will be outside each S_i'; here $\dot\cup$ denotes a disjoint union. Next, we have

$$\sum_{k=1}^{n} \binom{n}{k} k!(k+1)^{n-k} \le n! \sum_{k=1}^{n} \binom{n}{k} (n+1)^{n-k} \le n!(n+2)^n = 2^{O(n \log n)},$$

and the upper bound follows. □

Remark 1. The summation $\sum_{k=1}^{n} \binom{n}{k} k!(k+1)^{n-k}$ differs by one from Sloane's sequence A072597 [5]. These numbers are the coefficients of the exponential generating function of $1/(e^{-x} - x)$. It follows, by standard techniques, that these numbers are asymptotically given by $C_1 W(1)^{-n} n!$, where

$$W(1) \doteq .5671432904097838729999686622103555497538$$

is the Lambert W-function evaluated at 1, equal to the positive real solution of the equation $e^x = 1/x$, and C_1 is a constant, approximately

$$1.12511909098678593170279439143182676599.$$

The convergence is quite fast; this gives a somewhat more explicit version of the upper bound.

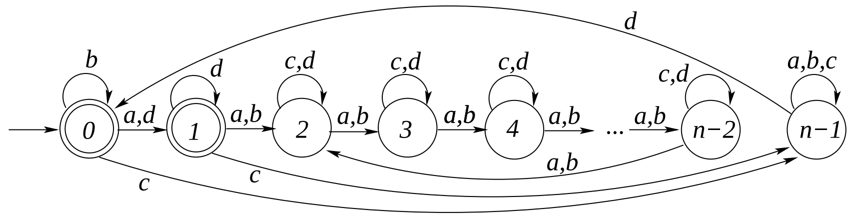

Fig. 5. DFA D over $\{a, b, c, d\}$ with many reachable states in DFA D_3 for L^{+c+}.

3 Lower Bound

We now turn to the matching lower bound on the state complexity of plus-complement-plus. The basic idea is to create one DFA where the DFA for L^{+c+} has many reachable states, and another where the DFA for L^{+c+} has many distinguishable states. Then we "join" them together in Corollary 2.

The following lemma uses a four-letter alphabet to prove the reachability of some specific states of the DFA D_3 for plus-complement-plus.

Lemma 3. *There exists an n-state DFA $D = (Q_n, \{a, b, c, d\}, \delta, 0, \{0, 1\})$ such that in the DFA D_3 for the language $L(D)^{+c+}$ every state of the form*

$$\Big\{\{0, q_1\} \cup S_1, \{0, q_2\} \cup S_2, \dots, \{0, q_k\} \cup S_k\Big\}$$

is reachable, where $1 \le k \le n-2$, $S_1, S_2, \dots, S_k$ are subsets of $\{2, 3, \dots, n-2\}$ with $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$, and the $q_1, \dots, q_k$ are pairwise distinct states in $\{2, 3, \dots, n-2\}$ that are not in S_k.

Proof. Consider the DFA D over $\{a, b, c, d\}$ shown in Fig. 5. Let L be the language accepted by the DFA D.

Construct the NFA N_1 for the language L^+ from the DFA D by adding loops on a and d in the initial state 0. In the subset automaton corresponding to the NFA N_1, every subset of $\{0, 1, \dots, n-2\}$ containing state 0 is reachable from the initial state $\{0\}$ on a string over $\{a, b\}$ since each subset $\{0, i_1, i_2, \dots, i_k\}$ of size k, where $1 \le k \le n-1$ and $1 \le i_1 < i_2 < \cdots < i_k \le n-2$, is reached from the set $\{0, i_2 - i_1, \dots, i_k - i_1\}$ of size $k-1$ on the string ab^{i_1-1}. Moreover, after reading every symbol of string ab^{i_1-1}, the subset automaton is always in a set that contains state 0. All such states are rejecting in the DFA D_2 for the language L^{+c}, and therefore, in the NFA N_3 for L^{+c+}, the initial state $\{0\}$ only goes to the rejecting state $\{0, i_1, i_2, \dots, i_k\}$ on ab^{i_1-1}.

Hence in the DFA D_3, for every subset S of $\{0, 1, \dots, n-2\}$ containing 0, the initial state $\{\{0\}\}$ goes to the state $\{S\}$ on a string w over $\{a, b\}$.

Now notice that transitions on symbols a and b perform the cyclic permutation of states in $\{2, 3, \dots, n-2\}$. For every state q in $\{2, 3, \dots, n-2\}$ and an integer i, let

$$q \ominus i = ((q - i - 2) \bmod n - 3) + 2$$

denote the state in $\{2,3,\ldots,n-2\}$ that goes to the state q on string a^i, and, in fact, on every string over $\{a,b\}$ of length i. Next, for a subset S of $\{2,3,\ldots,n-2\}$ let

$$S \ominus i = \{q \ominus i \mid q \in S\}.$$

Thus $S \ominus i$ is a shift of S, and if $q \notin S$, then $q \ominus i \notin S \ominus i$.

The proof of the lemma now proceeds by induction on k. To prove the base case, let S_1 be a subset of $\{2,3,\ldots,n-2\}$ and q_1 be a state in $\{2,3,\ldots,n-2\}$ with $q_1 \notin S_1$. In the NFA N_3, the initial state $\{0\}$ goes to the state $\{0\} \cup S_1$ on a string w over $\{a,b\}$. Next, state $q_1 \ominus |w|$ is in $\{2,3,\ldots,n-2\}$, and it is reached from state 1 on a string b^ℓ, while state 0 goes to itself on b. In the DFA D_3 we thus have

$$\big\{\{0\}\big\} \xrightarrow{a} \big\{\{0,1\}\big\} \xrightarrow{b^\ell} \big\{\{0, q_1 \ominus |w|\}\big\} \xrightarrow{w} \big\{\{0,q_1\} \cup S_1\big\},$$

which proves the base case.

Now assume that every set of size $k-1$ satisfying the lemma is reachable in the DFA D_3. Let

$$\mathcal{S} = \Big\{\{0,q_1\} \cup S_1, \{0,q_2\} \cup S_2, \ldots, \{0,q_k\} \cup S_k\Big\}$$

be a set of size k satisfying the lemma. Let w be a string, on which $\big\{\{0\}\big\}$ goes to $\big\{\{0\} \cup S_1\big\}$, and let ℓ be an integer such that 1 goes to $q_1 \ominus |w|$ on b^ℓ. Let

$$\mathcal{S}' = \Big\{\{0, q_2 \ominus |w| \ominus \ell\} \cup S_2 \ominus |w| \ominus \ell, \ldots, \{0, q_k \ominus |w| \ominus \ell\} \cup S_k \ominus |w| \ominus \ell\Big\},$$

where the operation $\ominus$ is understood to have left-associativity. Then $\mathcal{S}'$ is reachable by induction. On c, every set $\{0, q_i \ominus |w| \ominus \ell\} \cup S_i \ominus |w| \ominus \ell$ goes to the accepting state $\{n-1, q_i \ominus |w| \ominus \ell\} \cup S_i \ominus |w| \ominus \ell$ in the NFA N_3, and therefore also to the initial state $\{0\}$. Then, on d, every state $\{n-1, q_i \ominus |w| \ominus \ell\} \cup S_i \ominus |w| \ominus \ell$ goes to the rejecting state $\{0, q_i \ominus |w| \ominus \ell\} \cup S_i \ominus |w| \ominus \ell$, while $\{0\}$ goes to $\{0,1\}$. Hence, in the DFA D_3 we have

$$\begin{aligned}
\mathcal{S}' &\xrightarrow{c} \Big\{\{0\}, \{n-1, q_2 \ominus |w| \ominus \ell\} \cup S_2 \ominus |w| \ominus \ell, \ldots, \{n-1, q_k \ominus |w| \ominus \ell\} \cup S_k \ominus |w| \ominus \ell\Big\} \\
&\xrightarrow{d} \Big\{\{0,1\}, \{0, q_2 \ominus |w| \ominus \ell\} \cup S_2 \ominus |w| \ominus \ell, \ldots, \{0, q_k \ominus |w| \ominus \ell\} \cup S_k \ominus |w| \ominus \ell\Big\} \\
&\xrightarrow{b^\ell} \Big\{\{0, q_1 \ominus |w|\}, \{0, q_2 \ominus |w|\} \cup S_2 \ominus |w|, \ldots, \{0, q_k \ominus |w|\} \cup S_k \ominus |w|\Big\} \xrightarrow{w} \mathcal{S}.
\end{aligned}$$

It follows that $\mathcal{S}$ is reachable in the DFA D_3. This concludes the proof. $\square$

The next lemma shows that some rejecting states of the DFA D_3, in which no set is a subset of some other set, may be pairwise distinguishable. To prove the result it uses four symbols, one of which is the symbol b from the proof of the previuos lemma.

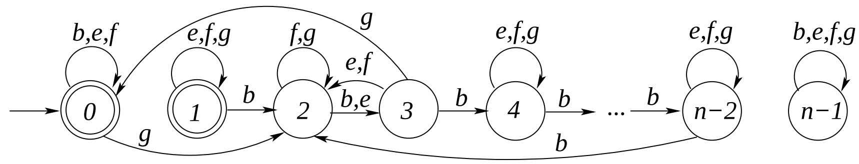

Fig. 6. DFA D over $\{b, e, f, g\}$ with many distinguishable states in DFA D_3.

Lemma 4. *Let $n \geq 5$. There exists an n-state DFA $D = (Q_n, \Sigma, \delta, 0, \{0, 1\})$ over a four-letter alphabet Σ such that all the states of the DFA D_3 for the language $L(D)^{+c+}$ of the form*

$$\Big\{\{0\} \cup T_1, \{0\} \cup T_2, \ldots, \{0\} \cup T_k\Big\},$$

in which no set is a subset of some other set and each $T_i \subseteq \{2, 3, \ldots, n-2\}$, are pairwise distinguishable.

Proof. To prove the lemma, we reuse the symbol b from the proof of Lemma 3, and define three new symbols e, f, g as shown in Fig. 6.

Notice that on states $2, 3, \ldots, n-2$, the symbol b performs a big permutation, while e performs a trasposition, and f a contraction. It follows that every transformation of states $2, 3, \ldots, n-2$ can be performed by strings over $\{b, e, f\}$. In particular, for each subset T of $\{2, 3, \ldots, n-2\}$, there is a string w_T over $\{b, e, f\}$ such that in D, each state in T goes to state 2 on w_T, while each state in $\{2, 3, \ldots, n-2\} \setminus T$ goes to state 3 on w_T. Moreover, state 0 remains in itself while reading the string w_T. Next, the symbol g sends state 0 to state 2, state 3 to state 0, and state 2 to itself.

It follows that in the NFA N_3, the state $\{0\} \cup T$, as well as each state $\{0\} \cup T'$ with $T' \subseteq T$, goes to the accepting state $\{2\}$ on $w_T \cdot g$. However, every other state $\{0\} \cup T''$ with $T'' \subseteq \{2, 3, \ldots, n-2\}$ is in a state containig 0, thus in a rejecting state of N_3, while reading $w_T \cdot g$, and it is in the rejecting state $\{0, 3\}$ after reading w_T. Then $\{0, 3\}$ goes to the rejecting state $\{0, 2\}$ on reading g.

Hence the string $w_T \cdot g$ is accepted by the NFA N_3 from each state $\{0\} \cup T'$ with $T' \subseteq T$, but rejected from any other state $\{0\} \cup T''$ with $T'' \subseteq \{2, 3, \ldots, n-2\}$.

Now consider two different states of the DFA D_3

$$\mathcal{T} = \big\{\{0\} \cup T_1, \ldots, \{0\} \cup T_k\big\},$$
$$\mathcal{R} = \big\{\{0\} \cup R_1, \ldots, \{0\} \cup R_\ell\big\},$$

in which no set is a subset of some other set and where each T_i and each R_j is a subset of $\{2, 3, \ldots, n-2\}$. Then, without loss of generality, there is a set $\{0\} \cup T_i$ in $\mathcal{T}$ that is not in $\mathcal{R}$. If no set $\{0\} \cup T'$ with $T' \subseteq T_i$ is in $\mathcal{R}$, then the string $w_{T_i} \cdot g$ is accepted from $\mathcal{T}$ but not from $\mathcal{R}$. If there is a subset T' of T_i such that $\{0\} \cup T'$ is in $\mathcal{R}$, then for each suset T'' of T' the set $\{0\} \cup T''$ cannot be in $\mathcal{T}$, and then the string $w_{T'} \cdot g$ is accepted from $\mathcal{R}$ but not from $\mathcal{T}$. $\square$

Corollary 2 (Star-Complement-Star: Lower Bound). *There exists a language L accepted by an n-state DFA over a seven-letter input alphabet, such that any DFA for the language L^{*c*} has $2^{\Omega(n \log n)}$ states.*

Proof. Let $\Sigma = \{a, b, c, d, e, f, g\}$ and L be the language accepted by n-state DFA $D = (\{0, 1, \dots, n-1\}, \Sigma, \delta, 0, \{0, 1\})$, where transitions on symbols a, b, c, d are defined as in the proof of Lemma 3, and on symbols d, e, f as in the proof of Lemma 4.

Let $m = \lceil n/2 \rceil$. By Lemma 3, the following states are reachable in the DFA D_3 for L^{+c+}:

$$\{\{0, 2\} \cup S_1, \{0, 3\} \cup S_2, \dots, \{0, m-2\} \cup S_{m-1}\},$$

where $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{m-1} \subseteq \{m-1, m, \dots, n-2\}$. The number of such subsets S_i is given by m^{n-m}, and we have

$$m^{n-m} \geq \left(\frac{n}{2}\right)^{\frac{n}{2}-1} = 2^{\Omega(n \log n)}.$$

By Lemma 4, all these states are pairwise distinguishable, and the lower bound follows. □

Hence we have an asymptotically tight bound on the state complexity of star-complement-star operation that is significantly smaller than 2^{2^n}.

Theorem 1. *The state complexity of star-complement-star is $2^{\Theta(n \log n)}$.* □

4 Applications

We conclude with an application.

Corollary 3. *Let L be a regular language, accepted by a DFA with n states. Then any language that can be expressed in terms of L and the operations of positive closure, Kleene closure, and complement has state complexity bounded by $2^{\Theta(n \log n)}$.*

Proof. As shown in [1], every such language can be expressed, up to inclusion of ε, as one of the following 5 languages and their complements:

$$L, L^+, L^{c+}, L^{+c+}, L^{c+c+}.$$

If the state complexity of L is n, then clearly the state complexity of L^c is also n. Furthermore, we know that the state complexity of L^+ is bounded by 2^n (a more exact bound can be found in [7]); this also handles L^{c+}. The remaining languages can be handled with Theorem 1. □

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